Second order conditions for free and constrained extrema

Free extrema[1]

First order conditions

The necessary conditions for finding an extremum are:

- $f'x|_{x_0,y_0}= 0$
- $f'y|_{x_0,y_0}= 0$

Second order conditions

The Hessian matrix for a two-variable function is defined as:

$$H = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$$

- if $|H|_{x_0,y_0}|> 0$, relative extremum
  - if $f''_{xx}(x_0, y_0) < 0$, relative maximum (negative definite Hessian)
  - if $f''_{xx}(x_0, y_0) > 0$, relative minimum (positive definite Hessian)
- if $|H|_{x_0,y_0}|< 0$, saddle point
- if $|H|_{x_0,y_0}|= 0$, uncertain case, there may be a maximum, minimum or no extrema.

Constrained extrema

When maximizing or minimizing functions subject to a constraint we construct the Lagrangian. Suppose we want to find the maximum or minimum of $f(x, y)$ subject to the following constraint: $g(x, y) = 0$. Then we construct the following function:

$$L(x, y, \lambda) = f(x, y) + \lambda[g(x, y)]$$

We obtain the first order conditions the same as in the case of free extrema:

- $L'x = 0$
- $L'y = 0$
- $L'\lambda = 0$

And for the second order conditions we construct the bordered Hessian:

$$\bar{H} = \begin{pmatrix} 0 & g'x & g'y \\ g'x & L''_{xx} & L''_{xy} \\ g'y & L''_{yx} & L''_{yy} \end{pmatrix}$$

And we evaluate the determinant of the bordered Hessian at the point and follow these 3 conditions:

- if $|\bar{H}|_{x_0,y_0}|> 0$, maximum
- if $|\bar{H}|_{x_0,y_0}|< 0$, minimum
- if $|\bar{H}|_{x_0,y_0}|= 0$, uncertain case, there may be a maximum, minimum or no extrema.

[1] All these conditions refer to relative maxima or minima for either free or constrained extrema

Examples

Free extremum 1

$$f(x,y) = z = xy + 1/x + 1/y$$

First order conditions:

$$f'x = y - x^{-2} = 0$$
$$f'y = x - y^{-2} = 0$$

Solving from the first equation:

$$y = x^{-2}$$

Substituting in the second one:

$$x - [x^{-2}]^{-2} = 0$$
$$x - x^4 = 0$$
$$x(1 - x^3) = 0$$

This leads me to the condition of $x = 1$ or $x = 0$. However $x = 0$ does not fulfill $y = x^{-2}$. But $x = 1$ leads us to $y = 1^{-2} = 1$.

$$f(1,1) = 3$$

Then we evaluate the point $(1, 1, 3)$ to see if it fulfills the sufficient conditions for maximum or minimum.

$$H|_{1,1}= \begin{pmatrix} 2x^{-3} & 1 \\ 1 & 2y^{-3} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

We calculate the determinant: $2*2-1*1 = 3 > 0$. Therefore we are at an extremum and as $f''_{xx}|_{1,1}= x^{-3} = 1 > 0$, we are at a minimum.

Free extremum 2

$$f(x,y) = z = (x-y)^4 + (y-1)^4$$

Calculate first order conditions:

$$f'x = 4(x-y)^3 = 0$$
$$f'y = -4(x-y)^3 + 4(y-1)^3 = 0$$

From the first equation I get that: $x = y$. Using this in the second equation:

$$-4(y-y)^3 + (y-1)^3 = 0$$
$$-4(0)^3 + 4(y-1)^3 = 0$$
$$(y-1)^3 = 0$$

Then $y = 1$ and therefore $x = 1$. Moving towards the second order conditions:

$$H|_{1,1}= \begin{pmatrix} 12(x-y)^2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 + 12(y-1)^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The determinant is 0. However, notice that the function can never take negative values as it is the sum of two positive powered terms. The minimum value it can take is 0. The point we found is precisely $(1, 1, 0)$. Therefore we are at a minimum.

Constrained Extremum

$$f(x, y) = x^2 + y^2$$

Subject to:

$$x = y$$

I construct the Lagrangian:

$$L = x^2 + y^2 + \lambda(y - x)$$

I calculate the necessary conditions:

$$L'x = 2x - \lambda = 0$$

$$L'y = 2y + \lambda = 0$$

$$L'\lambda = y - x = 0$$

From the first two equations, I obtain the value of λ:

$$\lambda = 2x$$

$$\lambda = -2y$$

I equate these and get: $-2y = 2x$, which means that $x = -y$. Substituting this into the third condition gives:

$$y + y = 0$$

Therefore $2y = 0$, which is only satisfied if $y = 0$, and this leads me to $x = 0$. Moving on to the second-order condition:

$$\bar{H}_{(0,0)} = \begin{pmatrix} 0 & -1 & 1 - 1 & 2 & 0\,1 & 0 & 2 \end{pmatrix}$$

Calculating the determinant:

$$0(22 - 00) + 1(-12 - 21) + 1(-10 - 21) = -2 < 0$$

We are dealing with a constrained minimum.